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GENERALIZED UNICORNS PROBLEM WITH A SPECIAL (α, β)-METRIC

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Abstract: In this paper, we study the generalized unicorns problem on regular (α, β)-metrics in the form of $F = \alpha\phi(s)$, $s = \beta/\alpha$, where α is a Riemannian metric and β is a 1-form on the manifold. We prove that, if $\phi = \phi(s)$ is a special polynomial in s, then F is a weak Landsberg metric if and only if F is a Berwald metric. Further, we prove that if $\phi = \phi(s)$ is a polynomial in s and F is not a Randers metric, then F is of relatively isotropic mean Landsberg curvature if and only if it is a Berwald metric.

Keywords and Phrases: Finsler space, (α, β)-metric, Berwald metric, weak Landsberg metric, generalized unicorns problem.

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1. Introduction

The unicorns problem is partially solved for an important class of Finsler (α, β)-metrics in the form of $F = \alpha\phi(s)$, $s = \beta/\alpha$, where α is a Riemannian metric and β is a 1-form on the manifold M. A Finsler metric F is called Landsberg metric if the Landsberg curvature $L := L_{ijk}dx^i \otimes dx^j \otimes dx^k$ vanishes [7]. A long existing open problem in Finsler geometry is
Is there any Landsberg metric which is not a Berwald metric ?

D. Bao [2] named the Landsberg metric that are not Berwald metric the unicorns. Z. Shen [12] has proved that a regular (α, β)-metric $F = \alpha\phi(s)$, $s = \beta/\alpha$, on a manifold M of dimension n $(n \geq 3)$ is a Landsberg metric if and only if F is a Berwald metric, i.e. there is no unicorn in regular (α, β)-metrics on the manifold M of dimension $n \geq 3$. On the other hand, Z. Shen and G. S. Asanov (see [1] and [11]) have constructed almost regular (α, β)-metrics which are Landsberg metrics but not Berwald metrics .
A Finsler metric F is called weak Landsberg metric if the mean Landsberg curvature $J = J_k dx^k$ vanishes, where $J_k := g^{ij} L_{ijk}$. It is clear that every Landsberg metric is a weak Landsberg metric. In [8], B. Li and Z. Shen have studied weak Landsberg (α, β)-metrics and characterized almost regular weak Landsberg (α, β)-metrics on an n-dimensional manifold M $(n \geq 3)$. They have also shown that [8] there exist almost regular weak Landsberg metrics which are not Landsberg metrics in dimension greater than two. At this juncture we have the following quetion:
Is there a regular weak Landsberg metric that is Berwald metric ?
The weak Landsberg metric that are not Berwald metric are called generalized unicorns.
In this paper, we mainly study the generalized unicorns problem for regular (α, β)-metrics.The main findings of this paper lies in Theorem 4.1 and Theorem 5.3. One can easily prove the following theorem:

Theorem 1.1. *Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be a regular (α, β)-metric on an n-dimensional manifold M $(n \geq 3)$, where α is a Riemannian metric and β is a 1-form on M. If $\phi = \phi(s)$ is a polynomial in s, then F is a weak Landsberg metric if and only if F is a Berwald metric.*

Theorem 1.1 is the generalization of the main theorem on unicorns problem for regular (α, β)-metrics given in [11]. It also gives a negative answer for the generalized unicorns problem on regular (α, β)-metrics in the case of the dimension $(n \geq 3)$.

Theorem 1.2. *Let $F = \alpha\phi(s), s = \frac{\beta}{\alpha}$, be a regular $(\alpha,\ \beta)$-metric of non-Randers type on an n-dimensional manifold M $(n \geq 3)$, where α is a Riemannian metric and β is a 1-form on M. If $\phi(s) = 1 + s + s^2 + s^3 + s^4 + ... + s^m$, $m \geq 2$, is a polynomial in s, then F is of relatively isotropic mean Landsberg curvature, $J + c(x)FI = 0$, if and only if it is a Berwald metric.*

2. Preliminaries

A Finsler manifold (M, F) is a C^∞- manifold M equipped with a Finsler metric which is a continuous function $F : TM \to [0, \infty)$ with the following properties:

1. F(x, y) is C^∞ on $TM\backslash\{0\}$. (Smoothness)

2. F(x, λ y) = λ F(x, y) for all $\lambda > 0$. (Positive Homogeneity)

3. The fundamental tensor $g_{ij}(x, y)$ is positive definite at all (x, y) $\in$ TM $\setminus\{0\}$, where

$$g_{ij}(x, y) := \frac{1}{2}[F^2]_{y^i y^j}(x, y).$$

The wellknown examples of Finsler metrics are Minkowski metrics, Riemannian metrics, Randers metric, Kropina metric, Matsumoto metric, square metric. The fundamental tensor is express as

$$g_y := g_{ij}(x, y) dx^i \otimes dx^j,$$

where $g_{ij} := \dfrac{1}{2}[F^2]_{y^i y^j}$.
The Cartan tensor is defined as

$$C_{ijk} := \frac{1}{4}[F^2]_{y^i y^j y^k} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}, \qquad I_i := g^{jk} C_{ijk},$$

where $(g^{ij}) = (g_{ij})^{-1}$. We define $C := C_{ijk} dx^i \otimes dx^j \otimes dx^k$, and $I := I_i(x, y) dx^i$, which are called Cartan torsion and the mean Cartan torsion. The Cartan torsion C and the mean Cartan torsion I both characterize the Riemannian metrics among Finsler metrics. A C^∞- curve $\sigma = \sigma(t)$ in a Finsler manifold (M, F) is called a geodesic if and only if it satisfies the following differential equation:

$$\ddot{\sigma}^i + 2G^i(\sigma(t), \ddot{\sigma}(t)) = 0,$$

where $G^i = G^i(x, y)$ are functions on TM defined by

$$G^i := \frac{1}{4} g^{il}\Big\{[F^2]_{x^k y^l} y^k - [F^2]_{x^l}\Big\}.$$

G^i are called the spray coefficients of F.
Let $\alpha = \sqrt{a_{ij}(x) y^i y^j}$ be a Riemannian metric and $\beta = b_i(x) y^i$, a 1-form on an n-dimensional manifold M. An (α, β)-metric is a Finsler metric express as

$$F = \alpha\phi(s), \;\; s = \frac{\beta}{\alpha},$$

where $\phi = \phi(s)$ is a continuous differential function on an open interval $(-b_0, b_0)$, satisfying

$$\phi(s) - s\phi^{'}(s) + (b^2 - s^2)\phi^{''}(s) > 0, \qquad |\, s \,| \le b < b_0, \tag{1}$$

where β satisfies $\parallel \beta_x \parallel_\alpha < b_0$, (or $\parallel \beta_x \parallel_\alpha \leq b_0$,) for any x $\in$ M. Such metrics are called regular (or almost regular) (α, β)-metrics ([9]).
Let G^i and G^i_α denote the spray coefficients of F and α respectively, given by ([11], [15])

$$G^i = G^i_\alpha + \alpha Q s^i_0 + \Theta\{-2\alpha Q s_0 + r_{00}\}\left\{\frac{y^i}{\alpha} + \frac{Q'}{Q - sQ'} b^i\right\}, \tag{2}$$

where

$$Q := \frac{\phi'}{\phi - s\phi'},$$

$$\Theta := \frac{Q - sQ'}{2[1 + sQ + (b^2 - s^2)Q']},$$

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}), \tag{3}$$

$$s^i_j := a^{im} s_{mj}, \quad r_j := b^m r_{mj}, \quad s_j := b_m s^m_j = b^m s_{mj}, \tag{4}$$

$$r_{00} := r_{ij} y^i y^j, \quad s^i_0 := s^i_j y^j.$$

Here "|" denotes the covariant derivative with respect to Levi-Civita Connection of α.
A Finsler metric F on a manifold M is called a Berwald metric if in any standard local coordinate system (x^i, y^i) in TM_0, the Christoffel symbol $\Gamma^i_{jk} = \Gamma^i_{jk}(x)$ are functions of $x \in M$ only, in which case, $G^i := \frac{1}{2}\Gamma^i_{jk}(x) y^j y^k$ are quadratic in $y = y^i \frac{\partial}{\partial x^i} |_x$. If F is a Berwald metric, the space (M, F) is called the Berwald space. It is wellknown that every Riemannian and locally Minkowskian metric are Berwald metric, i.e.,

$$\{Riemannian\} \ and \ \{locally\ Minkowskian\} \subset \{Berwald\}.$$

The landsberg curvature is defined by $L := L_{ijk} dx^i \otimes dx^j \otimes dx^k$. A Finsler metric F is called the Landsberg metric if Landsberg curvature L vanishes, i.e. L=0. Further, every Berwald metric is Landsberg metric but converse is not true, i.e.,

$$\{Berwald\ metrics\} \subset \{Landsberg\ metrics\}$$

but

$$\{Landsberg\, metrics\} \subsetneq \{Berwald\, metrics\}$$

always. There is a weaker non-Riemannian quantity than the Landsberg curvature L in Finsler geometry, $J = J_k dx^k$, where

$$J_k := g^{ij} L_{ijk},$$

where $(g_{ij})^{-1} = (g^{ij})$. More generally, we have the following

$$\begin{array}{ccc} \mathrm{C} & \longrightarrow & I : I_i = g^{jk} c_{ijk} \\ \downarrow & & \downarrow \\ L : L_{ijk} := C_{ijk|m} y^m & \longrightarrow & J : J_i := g^{jk} L_{ijk} = I_{i|m} y^m \end{array}$$

C : Cartan torsion I : mean Cartan torsion
L : Landsberg curvature J : mean Landsberg curvature.
Facts: F is Riemannian $\Leftrightarrow C = 0 \Leftrightarrow I = 0$.
A Finsler metric F is called the weak Landsberg metric if its mean Landsberg curvature J vanishes, i.e., J = 0. A Finsler metric F is said to be of relatively isotropic weak Landsberg curvature if F satisfies $J + cFI = 0$, where c = c(x) is a scalar function, I is mean Cartan torsion.
Clearly,

$$\{Landsberg\, metrics\} \subset \{weak\ Landsberg\, metrics\}$$
$$\subset \{Finsler\ metric\ satisfying\ J + cFI = 0\}.$$

Questions:

1. Is there a weak Landsberg metric which is not a Landsberg metric ?
2. Is there a regular weak Landsberg metric that is Berwald metric ?

The weak Landsberg metric that are not Berwald metric are called generalized unicorns.
By (2), it is easy to see that if β is parallel with respect to α, which is equivalent to $r_{ij} = s_{ij} = 0$, then $G^i = G^i_\alpha$. In this case, F is a Berwald metric.

Lemma 2.1. *For an (α,β)-metric $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$, $b = \| \beta_x \|_\alpha$, is a constant if and only if $r_i + s_i = 0$.*
We study the generalized unicorns problem for regular (α, β)-metrics. Now, we write some lemmas to explain it.

3. Some Important lemmas

Let $F = \alpha\phi(s)$, $s = \dfrac{\beta}{\alpha}$, be a regular ($\alpha$, β)-metric on a manifold M of dimension n ($n \geq 3$). For our aim, we need the following lemmas about the (α, β)-metrics of weak Landsberg type.

Lemma 3.1. *(See [8, 13]) Let $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$, be an (α, β)-metric on a manifold M of dimension n $(n \geq 3)$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M. Then F is a weak Landsberg metric, i.e., $J = 0$ if and only if β satisfies*

$$s_{ij} = 0, \tag{5}$$

$$r_{ij} = k(b^2 a_{ij} - b_i b_j) + e b_i b_j, \tag{6}$$

where $k = k(x)$ and $e = e(x)$ are scalar functions on M and $\phi = \phi(s)$ satisfies the following ODE:

$$\psi_1 k + es\psi_3 = 0, \tag{7}$$

where

$$\psi_1 := \sqrt{b^2 - s^2}\Delta^{1\backslash 2}\left[\frac{\sqrt{b^2 - s^2}\Phi}{\Delta^{\frac{3}{2}}}\right]^{'},$$

$$\psi_2 := 2(n+1)(Q - sQ^{'}) + 3\frac{\Phi}{\Delta},$$

$$\psi_3 := \frac{s}{b^2 - s^2}\psi_1 + \frac{b^2}{b^2 - s^2}\psi_2,$$

where

$$\Phi := -(n\Delta + 1 + sQ)(Q - sQ^{'}) - (b^2 - s^2)(1 + sQ)Q^{''},$$

$$\Delta := 1 + sQ + (b^2 - s^2)Q^{'}.$$

Using the lemma 3.1, we can prove the following lemma.

Lemma 3.2.*(See [4]) Let $F = \alpha\phi(s), s = \frac{\beta}{\alpha}$, be a non-Riemannian (α, β)-metric on a manifold M of dimension n $(n \geq 3)$. If F is a weak Landsberg metric, i.e., $J = 0$ and b is a constant on M, then F is a Berwald metric.*

Proof. By the assumptions that b is a constant and F is a non-Riemannian $(\alpha,$

β)-metric, we find that $b \neq 0$. Further, by lemma 2.1, we have $r_j + s_j = 0$. Then by (4) and (5), we have

$$r_j = 0. \tag{8}$$

Equation (6), contracting by b^i yields

$$r_j = eb^2 b_j. \tag{9}$$

From (8) and (9), we get

$$eb_j = 0. \tag{10}$$

Contracting this equation by b^j, we have $eb^2 = 0$, that implies e $= 0$ by $b \neq 0$, putting e $= 0$ in (7) yields

$$k\psi_1 = 0. \tag{11}$$

If $\psi_1 = 0$, then by the definition of ψ_1, we get $\left[\frac{\sqrt{b^2-s^2}\Phi}{\Delta^{\frac{3}{2}}}\right]' = 0$.
After solving this equation, we find that

$$\Lambda(s) := \frac{\sqrt{b^2 - s^2}\Phi}{\Delta^{\frac{3}{2}}}$$

is a constant for $\mid s \mid \leq b < b_0$. Letting s = b yields $\Lambda(s) = 0$, which implies that $\phi = 0$. By Proposition 2.2 in [13], we know that F is a Riemannian metric, which is a contradiction.
Therefore, we get k $= 0$ by (11), putting k $=$ e $= 0$ into (6), we get

$$r_{ij} = 0. \tag{12}$$

By (5) and (12), we see that β is a parallel with respect to α, which implies that F is a Berwald metric.

Lemma 3.3. *(See [4]) Let EQ denote the numerator of left of (7), then (7) holds if and only if*

$$EQ = 0, \tag{13}$$

holds and EQ can be expressed as below:

$$EQ = B_0 + b^2 B_2 + b^4 B_4 + b^6 B_6, \tag{14}$$

where

$b := \| \beta_x \|_\alpha,$

$B_0 := b_{01}s + b_{02}s^2 + b_{03}s^3 + b_{04}s^4 + b_{05}s^5 + b_{06}s^6 + b_{07}s^7 + b_{08}s^8,$

$B_2 := b_{20} + b_{21}s + b_{22}s^2 + b_{23}s^3 + b_{24}s^4 + b_{25}s^5 + b_{26}s^6,$

$B_4 := b_{40} + b_{41}s + b_{42}s^2 + b_{43}s^3 + b_{44}s^4,$

$B_6 := b_{60} + b_{61}s + b_{62}s^2,$

and

$b_{01} := 2(n+1)\phi^{'}\phi^5(e-k),$

$b_{02} := (n+1)\phi^4\big(2\phi^{''}\phi + 9\phi^{'^2}\big)(k-e),$

$b_{03} := 2\phi^3\big[(n+1)\phi\phi^{'}\phi^{''} + 8(n+1)\phi^{'^3} - (n+4)\phi^2\phi^{'''}\big](e-k),$

$b_{04} := \big[2\phi^{(4)}\phi^3 - (7n+24)\phi^2\phi^{'}\phi^{'''} + (n+25)\phi^2\phi^{''^2} - 8(n+1)\phi^{''}\phi^{'^2}\phi + 14(n+1)\phi^{'^4}\big]\phi^2(k-e),$

$b_{05} := \phi\big[3(n-4)\phi^3\phi^{''}\phi^{'''} - 3(3n+8)\phi^2\phi^{'^2}\phi^{'''} - 16(1+n)\phi\phi^{'^3}\phi^{''}$
$+(52+7n)\phi^2\phi^{'}\phi^{''^2} + 6(n+1)\phi^{'^5} + 6\phi^3\phi^{'}\phi^{(4)}\big](k-e),$

$b_{06} := \big[(4-5n)\phi^3\phi^{''^3} + 6(5+2n)\phi^{''^2}\phi^{'^2}\phi^2 + (1+n)\phi^{'^6} + 3\phi^{'''^2}\phi^4 - (8+5n)\phi^{'^3}\phi^2\phi^{'''}$
$-2\phi^4\phi^{''}\phi^{(4)} + (7n-24)\phi^3\phi^{'}\phi^{''}\phi^{'''} + 6\phi^{'^2}\phi^3\phi^{(4)} - 10(n+1)\phi^{'^4}\phi\phi^{''}\big](k-e),$

$b_{07} := \big[2(n+1)\phi^{'^5}\phi^{''} - (7n+4)\phi^{'^3}\phi^{''^2}\phi + 4\phi^{'}\phi^3\phi^{''}\phi^{(4)} - 2\phi^{'^3}\phi^2\phi^{(4)} - 6\phi^{'}\phi^{'''^2}\phi^3$
$+4(2n-1)\phi^2\phi^{'}\phi^{''^3} + (n-2)\phi^3\phi^{''^2}\phi^{'''} + n\phi\phi^{'^4}\phi^{'''} + (12-5n)\phi\phi^{'^2}\phi^{''}\phi^{'''}\big](k-e),$

$b_{08} := \big[(n+1)\phi^{'^4}\phi^{''^2} + 2(n-2)\phi^2\phi^{''^4} + n\phi\phi^{'^3}\phi^{''}\phi^{'''} - 3n\phi\phi^{'^2}\phi^{''^3}$
$+3\phi^2\phi^{'^2}\phi^{'''^2} - 2\phi^2\phi^{'^2}\phi^{''}\phi^{(4)} + (2-n)\phi^2\phi^{'}\phi^{''^2}\phi^{'''}\big](k-e),$

$b_{20} := -(n+1)k\phi^{'^2}\phi^4,$

$b_{21} := 2\phi^3\big[3e\phi^2\phi^{'''} - (4+n)k\phi^2\phi^{'''} + 2(n+1)(e-k)\phi\phi^{'}\phi^{''} + 2(n+1)k\phi^{'^3}\big],$

$b_{22} := -\phi^2\{2(2k-e)\phi^3\phi^{(4)} + [(n+31)k + 4e(n-5)]\phi^{''^2}\phi^2 + 6k(n+1)\phi^{'^4}$
$+2(n+1)(7e-8k)\phi^{''}\phi^{'^2}\phi + [(18+n)e - 8(n+3)k]\phi^{'}\phi^2\phi^{'''}\},$

$b_{23} := \phi\{3(2-n)e\phi^3\phi^{''}\phi^{'''} + [3e(n+6) - 12(n+2)k]\phi^{'^2}\phi^2\phi^{'''} + 4(n+1)k\phi^{'^5} + 6(n-4)k\phi^{''}\phi^3\phi^{'''}$
$+[6(2n+11)k + e(n-44)]\phi^2\phi^{'}\phi^{''^2} + 6(n+1)(3e-4k)\phi\phi^{'^3}\phi^{''} + 6(2k-e)\phi^{'}\phi^3\phi^{(4)}\},$

$b_{24} := [4e(2n-3) + 3(16-5n)k]\phi^3\phi^{'}\phi^{''}\phi^{'''} + 6(e-2k)\phi^{'^2}\phi^3\phi^{(4)} - k(1+n)\phi^{'^6}$
$+2(n+1)(8n-5e)\phi^{'^4}\phi\phi^{''} + 6[e(2n+5) - (4n+7)k]\phi^{'^2}\phi^{''^2}\phi^2 + [8k(n+1) - 3e(n+2)]\phi^{'^3}\phi^2\phi^{'''} + 2(3k-2e)\phi^4\phi^{''}\phi^{(4)} + [2(5n-1)k - 3(n+6)e]\phi^3\phi^{''^3} - 3(2e-3k)\phi^{'''^2}\phi^4,$

$b_{25} := 2(e-2k)(n+1)\phi^{'^5}\phi^{''} + [12k(n-2) + e(6-7n)]\phi^{'^2}\phi^2\phi^{''}\phi^{'''} + n(16k-11e)\phi^{'^3}\phi^{''^2}\phi$
$+2(2k-e)\phi^{'^3}\phi^{(4)}\phi^2 + 2(5k-4e)\phi^{'^3}\phi^{''^2}\phi + 6(3k-2e)\phi^{'}\phi^{'''^2}\phi^3 + 4(2e-3k)\phi^{'}\phi^3\phi^{''}\phi^{(4)}$
$+n(e-2k)\phi\phi^{'''}\phi^{'^4} + [2(9e+k) + n(9e-19k)]\phi^{'}\phi^2\phi^{''^3} + (2-n)(3k-2e)\phi^3\phi^{'''}\phi^{''^2},$

$b_{26} := \{2(n-2)\phi^2\phi^{''^4} + (n+1)\phi^{'^4}\phi^{''^2} - 3n\phi^{'^2}\phi^{''^3}\phi + n\phi\phi^{''}\phi^{'''}\phi^{'^3}$
$+3\phi^{'^2}\phi^2\phi^{'''^2} - 2\phi^{'^2}\phi^2\phi^{''}\phi^{(4)} + (2-n)\phi^2\phi^{'}\phi^{''^2}\phi^{'''}\}(2e-3k),$

$b_{40} := k\phi^3[6\phi\phi^{''^2} - n\phi\phi^{'}\phi^{'''} + 2\phi^2\phi^{(4)} - 2(n+1)\phi^{'^2}\phi^{''}],$

$b_{41} := \phi^2\{3[(4-n)k+2e]\phi^2\phi^{''}\phi^{'''} + [2(n+1)e - k(14+5n)]\phi\phi^{'}\phi^{''2} - 6k\phi^2\phi^{'}\phi^{(4)}$
$+3nk\phi\phi^{'2}\phi^{'''} + 6k(n+1)\phi^{'3}\phi^{''}\}$,
$b_{42} := -\phi\{2(3k-e)\phi^3\phi^{''}\phi^{(4)} + 3(e-3k)\phi^3\phi^{'''2} - 6k\phi^{'2}\phi^2\phi^{(4)} + [(5n+8)k + 2(n-11)e]\phi^2\phi^{''3} + 3nk\phi^{'3}\phi\phi^{'''} + (5e-13k)(n+1)\phi\phi^{'2}\phi^{''2} + 6k(n+1)\phi^{'4}\phi^{''} + [3(8-3n)k + (12+n)e]\phi^2\phi^{'}\phi^{''}\phi^{'''}\}$,
$b_{43} := \{[2k(4+7n) - e(n+22)]\phi^2\phi^{'}\phi^{''3} + (n-2)(3k-e)\phi^3\phi^{''2}\phi^{'''} + [4e(n+1) - k(8+11n)]\phi^{'3}\phi^{''2}\phi + 6(e-3k)\phi^{'}\phi^{'''2}\phi^3 + 4(3k-e)\phi^{'}\phi^3\phi^{''}\phi^{(4)} + 2k(n+1)\phi^{'5}\phi^{''} - 2k\phi^{'3}\phi^2\phi^{(4)} + [2e(n+3) + 3k(4-3n)]\phi^{'2}\phi^2\phi^{''}\phi^{'''} + nk\phi\phi^{'4}\phi^{'''}\}$,
$b_{44} := \{2(n+2)\phi^2\phi^{''4} - 2\phi^{'2}\phi^2\phi^{''}\phi^{(4)} + n\phi\phi^{'3}\phi^{''}\phi^{'''} + (2-n)\phi^{'}\phi^2\phi^{''2}\phi^{'''}$
$+(n+1)\phi^{'4}\phi^{''2} - 3n\phi^{'2}\phi^{''3}\phi + 3\phi^{'2}\phi^{'''2}\phi^2\}(3k-e)$,
$b_{60} := -k\phi^2[3\phi^2\phi^{'''2} - 2\phi^2\phi^{''}\phi^{(4)} - 6\phi\phi^{''3} + n\phi\phi^{'}\phi^{''}\phi^{'''} + (1+n)\phi^{''2}\phi^{'2}]$,
$b_{61} := k\phi\{-4\phi^2\phi^{'}\phi^{''}\phi^{(4)} + 6\phi^2\phi^{'''2}\phi^{'} + (2-n)\phi^2\phi^{''2}\phi^{'''} - 3(2+n)\phi\phi^{'}\phi^{''3}$
$+2n\phi\phi^{'2}\phi^{''}\phi^{'''} + 2(n+1)\phi^{'3}\phi^{''2}\}$,
$b_{62} := -k\{2(n-2)\phi^2\phi^{''4} - 2\phi^{'2}\phi^2\phi^{''}\phi^{(4)} - 3n\phi^{'2}\phi^{''3}\phi + 3\phi^{'2}\phi^{'''2}\phi^2$
$+(n+1)\phi^{'4}\phi^{''2} + (2-n)\phi^{'}\phi^2\phi^{''2}\phi^{'''} + n\phi\phi^3\phi^{''}\phi^{'''}\}$.

4. Special unicorns related with a polynomial

Assume that $\phi = \phi(s)$ is a polynomial which is expressed as below:

$$\phi = 1 + s + s^2 + s^3. \tag{15}$$

Lemma 4.1. *Let $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$, be a non-Riemannian weak Landsberg (α, β)-metric on a manifold M of dimension n $(n \geq 3)$ and $\phi = \phi(s) = 1 + s + s^2 + s^3$, a polynomial in s. Then b is a nonzero constant on M. In this case, F is a Berwald metric.*

Proof. Firstly, let F be a non-Riemannian (α, β)-metric, i.e., any point $x \in M$ satisfying b(x) = 0 must be a isolated point on M. Then, if a continuous function $\mu = \mu(x)$ satisfies $b(x)\mu(x) = 0$ on M, then $\mu(x) \equiv 0$ on M by the continuity of $\mu = \mu(x)$.

Now, by lemma 3.1 and 3.3, F= $\alpha\phi(s)$, s= β/α, is a weak Landsberg metric if and only if ϕ satisfies (13) and β satisfies (5) and (6). Consider following three cases:

Case A. When $deg(\phi) = 1$, i.e. $\phi = 1 + s$. In this case, the metric F becomes Randers metric.

Putting $\phi(s) = 1 + s$ in equation (13) and by using the Maple program, we have

$$a_2 s^2 + a_1 s + a_0 = 0, \tag{16}$$

where

$$a_0 := -(n+1)kb^2,$$

$$a_1 := 2(n+1)(e-k),$$

$$a_2 := (n+1)(e-k).$$

Note that a_0, a_1, a_2 are independent of s, so (16) is equivalent to $a_0 = a_1 = a_2 = 0$. It is easy to see that $k = e = 0$, from (6), we eliminate $r_{ij} = 0$. By (5) and $r_{ij} = 0$, we obtain that β is parallel with respect to α. In this case, b is a constant and F is a Berwald metric, i.e., Randers metric becomes Berwald metric.

Case B. When $deg(\phi) = 2$, i.e., $\phi(s) = 1 + s + s^2$. In this case, metric F reduces to first approximate Matsumoto metric, i.e., $F = \alpha + \beta + \dfrac{\beta^2}{\alpha}$.

Putting $\phi(s) = 1 + s + s^2$ in equation (13) and using the Maple program, we get

$$\eta_i s^i = 0, \quad 0 \le i \le 11, \tag{17}$$

where $\eta_0, \eta_1, ..., \eta_{11}$ are all independent of s. Particularly, we have

$$\eta_0 = -b^2 k(1+2b^2)(2b^2(n-11)+(n+1)), \tag{18}$$

$$\begin{aligned}\eta_2 = {} & 8(19-14n)kb^6 + 4b^4[(49+n)e-(14+23n)k] \\ & +4b^2[(25+n)e-5(n+7)k]+(n+1)(e-k),\end{aligned} \tag{19}$$

$$\eta_{10} = 3(k-e)(41n-67) + 24b^2(2e+k)(2-n), \tag{20}$$

$$\eta_{11} = 18(n-1)(k-e). \tag{21}$$

From $\eta_{11} = 0$, we get $k = e$. From equation (20), we get

$$\eta_{10} = 72(2-n)b^2 e,$$

If $\eta_{10} = 0$, then $k = e = 0$. Plugging $\phi = 1 + s + s^2$ into (1) and letting $s = 0$ yields $1 + 2b^2 > 0$. Thus $\eta_0 = 0$ implies that $k = 0$ by equation (18). Putting $k = 0$ into equation (19) yields

$$\eta_2 = [4b^4(49+n) + 4b^2(25+n) + (n+1)]e.$$

By $\eta_2 = 0$, we have $e = 0$.

Similar to proof of case A, we have that b is a constant and F is a Berwald metric.

Thus, every first approximate Matsumoto metric reduces to a Berwald metric.
Case C. When $deg(\phi) = 3$, i.e., $\phi = 1 + s + s^2 + s^3$. In this case, metric F reduces to second approximate Matsumoto metric, i.e., $F = \alpha + \beta + \dfrac{\beta^2}{\alpha} + \dfrac{\beta^3}{\alpha^2}$.
Putting $\phi(s) = 1 + s + s^2 + s^3$ in equation (13) and using the Maple program, we get

$$f_i s^i = 0, \quad 0 \leq i \leq 17, \tag{22}$$

where $f_0, f_1, ..., f_{17}$ are all independent of s. Particularly, we have

$$f_{13} = 2(kf_{13k} + ef_{13e}), \tag{23}$$

$$f_{14} = kf_{14k} + ef_{14e}, \tag{24}$$

$$f_{15} = -2(kf_{15k} + ef_{15e}), \tag{25}$$

$$f_{16} = -4(kf_{16k} + ef_{16e}), \tag{26}$$

$$f_{17} = 192(n-1)(k-e), \tag{27}$$

where
$f_{13k} := 24(53n - 150)b^4 + 12(2217 - 919n)b^2 + (11294n - 16861)$,
$f_{13e} := 48(60n - 189)b^4 + (13809 - 4846n)b^2 + (16867 - 11288n)$,
$f_{14k} := 576(n-3)b^4 + 12(1533 - 761n)b^2 + 3(4699n - 8511)$,
$f_{14e} := 1728(n-3)b^4 + 12b^2(2364 - 1045n) + (26557 - 14097n)$,
$f_{15k} := (69n - 133)b^2 + (6252 - 3153n)$,
$f_{15e} := 24(159n - 331))b^2 + (3153n - 6252)$,
$f_{16k} : 192(n-2)b^2 + (543 - 366n)$,
$f_{16e} := 576(n-2)b^2 + (366n - 543)$.
By equation (27), $f_{17} = 0$, means that $(k - e) = 0$, i.e., $k = e$. Putting $k = e$ in equation (26), we get

$$f_{16} = -3072(n-2)b^2 e.$$

From the equation $f_{16} = 0$, we get $e = 0$. Thus we have $k = e = 0$.
Similar to proof of case A, we have that b is a constant and F is a Berwald metric. Thus, in this case, every second approximate Matsumoto metric becomes a Berwald metric.

5. $(\alpha,\ \beta)$-metrics with relatively isotropic weak Landsberg curvature

In this section, we study about regular $(\alpha,\ \beta)$-metrics of non-Randers type with relatively isotropic mean Landsberg curvature. Here, we must mention the following lemmas:

Lemma 5.1. *(See [13]) Let $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$, be a regular (α, β)-metric on a manifold M of dimension n $(n \geq 3)$. Then F is of relatively isotropic mean Landsberg curvature, i.e., there exist a scalar function $c = c(x)$ on M such that $J + c(x)FI = 0$, if and only if β satisfies*

$$s_{ij} = 0, \tag{28}$$

$$r_{ij} = k(b^2 a_{ij} - b_i b_j) + e b_i b_j, \tag{29}$$

where $k = k(x)$ and $e = e(x)$ are scalar functions on M and $\phi = \phi(s)$ satisfies the following ODE:

$$\{\psi_1 k + es\psi_3\} + c(x)\Phi(\phi - s\phi^{'}) = 0, \tag{30}$$

where Φ, ψ_1, ψ_3 are defined in section 3.

Lemma 5.2. *(See [5]) Let PPE denote the numerator of the left of equation (30), then (30) holds if and only if*

$$PPE = 0, \tag{31}$$

and PPE can be expressed as

$$PPE := EQ + PE, \tag{32}$$

where EQ is defined by equation (14) and PE can be written as

$$PE := (b^2 - s^2)\phi(\phi - s\phi^{'})\phi^{'''} + \{(n-2)(s^2 - b^2)s\phi\phi^{''2}$$
$$+(n+1)(\phi - s\phi^{'})[(b^2 - s^2)\phi^{'} - s\phi]\phi^{''} + (n+1)(\phi - s\phi^{'})^2\phi\}.$$ We prove the following theorem:

Theorem 5.3. *Let $F = \alpha\phi(s), s = \frac{\beta}{\alpha}$, be a regular $(\alpha,\ \beta)$-metric of non-Randers type on an n-dimensional manifold M $(n \geq 3)$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M. If $\phi(s) = 1+s+s^2+s^3+s^4+...+s^m$,*

$m \geq 2$, is a polynomial in s, then F is of relatively isotropic mean Landsberg curvature, $J + c(x)FI = 0$, if and only if it is a Berwald metric.

Proof. Firstly, let F is of relatively isotropic mean Landsberg curvature. By (31) and (32), lemma 5.1 and lemma 5.2 holds, Express $\phi(s)$ as below:

$$\phi = 1 + s + s^2 + s^3 + s^4 + ... + s^m, \quad m \geq 2. \tag{33}$$

From (33), (32) and (31), we have

$$v_i s^i = 0, \quad 0 \leq i \leq r, \tag{34}$$

where $v_0, v_1, ..., v_r$ are independent of s.
Now, using the Maple program, we get the following results in three steps.
Step 1. When $m = 2$, $\phi = 1 + s + s^2$, we get

$$v_r = -216nc,$$

and $r = 13$. In this case, because $v_r = 0$, so c must be zero.
Step 2. When $m = 3$, $\phi = 1 + s + s^2 + s^3$, we get

$$v_r = -12288cn,$$

and $r = 20$. In this case, because $v_r = 0$, so c must be zero.
Step 3. When $m \geq 4$, we have

$$v_r = -4nm(m + 1)^3(m - 1)^4 c,$$

and

$$r = 7m - 1.$$

By the same reason as above, c must also be zero. Therefore we conclude that, if $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$ is of relatively isotropic mean Landsberg curvature, then F is a weak Landsberg metric, which implies that F is Berwald metric. In other words, if Randers metric is of relatively isotopic mean Landsberg curvature then it is a weak Landsberg metric and consequently becomes a Berwald metric. Similar, results hold for first and second approximate Matsumoto metric.

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